Defn) Closed Transformation: A linear transformations T which maps a subspace D of a Banach space B into a Banach space B' is called closed if its graph $G(T) = \{(x, Tx) : x \in D\}$ is closed in $B \times B'$.

Hence, T is closed iff $x_n \to x$ & $Tx_n \to y$ imply $x \in D$, $y = Tx$. In this language the closed graph theorem may be stated as follows- If B & B' are Banach spaces and if T is a linear transformation of B into B' then T is continuous iff T is closed.

S. Paper M.U. M.Sc. 95

Q.No.> Construct an example of a linear transformation which is closed but not continuous (i.e. Example of a closed but unbounded linear transformation).

Soln. Let $X = Y = C[0,1]$ with the sup norm. Then the subspace C_1 of X consisting of all continuously differentiable functions constitute a natural domain on which the differential operator $\frac{d}{dt}$ defines a linear transformation from C_1 into Y. This linear transformation is unbounded but closed.

Here $C_1 = \{f \in X : f' \in Y\}$. Put $T(f) = \frac{df}{dt} = f'$ for $f \in C_1$. Then $T: C_1 \to Y$ is a linear transformation. T is unbounded for if $f_n(t) = t^n$, $n = 1, 2, 3, \ldots$ where $t \in [0,1]$, then $f_n'(t) = nt^{n-1}$,

and thus, $\|f_n\| = \sup\limits_{t \in [0,1]} |f_n(t)| = 1$.

While $\|Tf_n\| = \|f_n'\| = \sup\limits_{t \in [0,1]} |f_n'(t)| = n \to \infty$.

This shows that while (f_n) is bounded, (Tf_n) is unbounded. Therefore, T is an unbounded linear transformation.

We now show that T is a closed transformation. Let $f_n \in C_1$ be such that $\lim\limits_{n\to\infty} f_n = f$ & $\lim\limits_{n\to\infty} Tf_n = \lim\limits_{n\to\infty} \frac{df_n}{dt} = y$ where the convergence is uniform.

$$\text{Then, } \int_0^t y(s)\,ds = \int_0^t \lim_{n\to\infty} \frac{df_n}{dt}\,ds$$

$$= \lim_{n\to\infty} \int_0^t \frac{df_n}{dt}\,ds = \lim_{n\to\infty} [f_n(t) - f_n(0)]$$

$$= f(t) - f(0).$$

Thus, $f(t) = f(0) + \int_0^t y(s)\,ds$.

Hence, it follows that $f \in C_1$ & $\frac{df}{dt} = y$ i.e. $Tf = y$.

Therefore, T is a closed transformation.

M.U

Q.No. Let N & N' be two normed linear spaces. Let $B(N, N')$ be the set of all continuous linear transformations from N to N'. Then show that

(i) $B(N, N')$ is a normed linear spaces.

(ii) $B(N, N')$ is a Banach space if N' is a Banach space.

Proof (i) Let $T_1, T_2 \in B(N, N')$ & $\alpha T_1 \in B(N, N')$ let (x_m) be a bounded sequence in N. Then, there exists a subsequence (x_{m_j}) such that $(T_1(x_{m_j}))$ and $(T_2(x_{m_j}))$ converge in N'. Then, clearly $(\alpha T_1(x_{m_j}))$ and $((T_1 + T_2)(x_{m_j}))$ converge in N'. Hence αT_1 & $T_1 + T_2 \in B(N, N')$. It can be easily seen that $B(N, N')$ is a normed linear spaces.

(ii) Let N' be a Banach space. In order to show that $B(N, N')$ is a Banach space it is sufficient to show that $B(N, N')$ is a closed subspace of the Banach space $B(N, N')$. Let (T_n) be a sequence in $B(N, N')$ & let $T_n \to T \in B(N, N')$.

In order to show that T is compact it is sufficient to show that $T(S)$ is totally bounded where S is the open unit ball in X. Let $\epsilon > 0$ be given and choose +ve integer n_0 such that $\|T - T_{n_0}\| < \epsilon$.

Since $T_{n_0}(S)$ is totally bounded, there exists $x_1, \ldots, x_m \in S$ such that,

$$T_{n_0}(S) \subseteq \bigcup_{K=1}^{m} S_{\epsilon}(T_{n_0}(x_K))$$

Now, if $x \in S$, then $\|T(x) - T_{n_0}(x)\| < \epsilon$, also there exists x_K, $1 \le K \le m$ such that $\|T_{n_0}(x) - T_{n_0}(x_K)\| < \epsilon$. Therefore,

$$\|T(x) - T(x_K)\| \le \|T(x) - T_{n_0}(x)\| + \|T_{n_0}(x) - T_{n_0}(x_K)\| + \|T_{n_0}(x_K) - T(x_K)\|$$
$$< 3\epsilon$$

Thus, $T(S) \subseteq \bigcup_{K=1}^{m} S_{3\epsilon}(T(x_K))$.

Since, this true for every $\epsilon > 0$, $T(S)$ is totally bounded Hence T is Compact. Thus, $T \in B(N, N')$. Thus every sequence in $B(N, N')$ Converges in $B(N, N')$. Hence $B(N, N')$ is a closed subspace of the Banach space $B(N, N')$. Therefore, $B(N, N')$ is a Banach space.

QNo→ For normed linear spaces X and Y, the mapping $T \to T^*$ of $B(X, Y)$ into $B(Y^*, X^*)$ has the following properties,

(a) $(\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^*$.

(b) $(T_1 \cdot T_2)^* = T_2^* T_1^*$.

Proof – (a) We have to show that,

$$(\alpha T_1 + \beta T_2)^*(g) = (\alpha T_1^* + \beta T_2^*)(g) \text{ for every } g \in Y^*.$$

i.e. to show that,

$$[(\alpha T_1 + \beta T_2)^*(g)](x).$$

$= [(\alpha T_1^\star + \beta T_2^\star)(g)](x)$

for every $g \in Y^\star$ and each $x \in X$.

Now, $[(\alpha T_1 + \beta T_2)^\star (g)](x) = g[(\alpha T_1 + \beta T_2)(x)]$

$= g[\alpha T_1(x) + \beta (T_2(x))]$

$= \alpha g(T_1(x)) + \beta g(T_2(x))$.

$= \alpha T_1^\star (g(x)) + \beta T_2^\star (g(x))$

$= (\alpha T_1^\star + \beta T_2^\star) g(x)$.

$= [(\alpha T_1^\star + \beta T_2^\star) g](x)$.

Hence, (a) is valid.

(b) For each $g \in Y^\star$ & $x \in X$, we have

$[(T_1 T_2)^\star (g)](x) = g((T_1 . T_2)(x))$

$= g(T_1(T_2(x))) = [T_1^\star (g)](T_2(x))$.

$= [T_2^\star (T_1^\star (g))](x) = [(T_2^\star T_1^\star)(g)](x)$

Hence, $(T_1 . T_2)^\star = T_2^\star T_1^\star$.